

A CAUCHY-SCHWARZ INEQUALITY FOR REPRESENTATIONS OF $SU(2)$

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ABSTRACT. Motivated by some applications to calculating order of poles of certain (local or global) L -functions, the author considers a Cauchy-Schwarz type inequality for representations of $SU(2)$.

1.

For any finite dimensional representation E of $SU(2)$, let $E[i]$ denote the subspace of E on which \mathbb{S}^1 , the standard maximal torus in $SU(2)$, operates by the character $z \rightarrow z^i$.

Let V and W be two finite dimensional representations of SU_2 . This note tries to estimate

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) - 2 \dim(V \otimes W[1]),$$

or, a closely related function:

$$d(V, W) = \dim(V \otimes V[2] \oplus W \otimes W[2]) - 2 \dim(V \otimes W[1]).$$

Since the space of finite dimensional (virtual) representations of $SU(2)$ is itself a free abelian group with a natural basis consisting of the isomorphism classes of irreducible representations of $SU(2)$, the question about $d(V, W)$ amounts to one regarding:

$$d(V, W) = q(V) + q(W) - 2B(V, W),$$

but notice that in the above expression for $d(V, W)$, the quadratic form q and the bilinear form B are different, and as a result there seems no natural context in which $d(V, W)$ may be studied. Since q is a positive definite quadratic form, and B is closely related to its associated bilinear form, $d(V, W)$ is typically non-negative, say for $\dim V - \dim W = 1$, but not always; one of the motivating question has been if we can classify the pairs (V, W) for which $d(V, W)$ is negative, or with $d(V, W) = 0$? We attempt some of this here, but without having any road-map, we seem rather lost.

For any finite dimensional representation E of $SU(2)$, write $E = E^+ \oplus E^-$ as the sum of the subspaces on which the center of $SU(2)$ operates trivially, and non-trivially; thus for representations V, W , write,

$$\begin{aligned} V &= V^+ \oplus V^-, \\ W &= W^+ \oplus W^-. \end{aligned}$$

Since,

$$\begin{aligned} V \otimes V[0] \oplus W \otimes W[0] &= V^+ \otimes V^+[0] \oplus V^- \otimes V^-[0] \oplus W^+ \otimes W^+[0] \oplus W^- \otimes W^-[0] \\ V \otimes W[1] &= V^+ \otimes W^-[1] \oplus V^- \otimes W^+[1], \end{aligned}$$

it follows that,

$$\begin{aligned} V \otimes V[0] + W \otimes W[0] - 2V \otimes W[1] &= V^+ \otimes V^+[0] + W^- \otimes W^-[0] - 2V^+ \otimes W^-[1] \\ &+ V^- \otimes V^-[0] + W^+ \otimes W^+[0] - 2V^- \otimes W^+[1]. \end{aligned}$$

Therefore to understand the behavior of the function

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) - 2 \dim(V \otimes W[1]),$$

it suffices to assume that the center of $SU(2)$ operates by scalar on V and on W , and these scalars are different.

Lemma 1. *Let V and W be two finite dimensional representations of SU_2 such that the center of $SU(2)$ operates by scalars on V and on W . Then we have the estimates,*

(1) *If the central characters of V and W are the same,*

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) \geq 2 \dim(V \otimes W[0]);$$

further, the inequality becomes an equality if and only if $V = W$. (This part of the lemma remains true for the zero weight space for a maximal torus in any compact Lie group.)

(2) *If the central characters of V and W are opposite,*

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) \geq 2 \dim(V \otimes W[1]);$$

further, the inequality is never an equality.

Proof. Let χ_1 (resp. χ_2) denote the character of the representation V (resp. W). The first part of the lemma follows by integrating on \mathbb{S}^1 the inequality,

$$(\chi_1 - \chi_2)^2 = \chi_1^2 + \chi_2^2 - 2\chi_1\chi_2 \geq 0.$$

For the proof of the second part of the lemma, let St_i denote the character of the standard $(i + 1)$ -dimensional irreducible representation of SU_2 , i.e. $\text{Sym}^i(\mathbb{C}^2)$, which is a real valued function on SU_2 . By the Clebsch-Gordon theorem, St_i^2 is the character of the sum of all irreducible representations of SU_2 with multiplicity one of highest weights $\leq 2i$, and of trivial central character, and similarly $St_i \cdot St_{i-1}$ is the character of the sum of all irreducible representations of SU_2 with multiplicity one of highest weights $\leq 2i - 1$, and of non-trivial central character. It follows from the representation theory of SU_2 that any irreducible representation of SU_2 with trivial central character contains the trivial character of \mathbb{S}^1 with multiplicity one, whereas any irreducible representation of SU_2 with non-trivial central character contains the character $z \rightarrow z$ of \mathbb{S}^1 with multiplicity one. Integrating the following inequality for n sufficiently large (this time on SU_2 on which we will fix a Haar measure giving it volume 1)

$$(\chi_1 St_n - \chi_2 St_{n-1})^2 = \chi_1^2 St_n^2 + \chi_2^2 St_{n-1}^2 - 2\chi_1\chi_2 St_n St_{n-1} \geq 0,$$

and interpreting the various terms using the Schur orthogonality, and the facts on representation theory of SU_2 just recalled, the inequality in the second part of the lemma follows.

To see that the inequality in the second part of the Lemma can never be an equality, note that the equality can happen only if $\chi_1 St_n - \chi_2 St_{n-1} = 0$. But by the same argument, we must then also have, $\chi_1 St_{n-1} - \chi_2 St_n = 0$. These two

identities mean that $\chi_1 = \pm\chi_2$, which is not an option because of different central characters. \square

Corollary 1. *Let $V = \sum_{i=0}^{i=d} n_i \text{Sym}^{2i}(\mathbb{C}^2)$, and $W = \sum_{i=1}^{i=d} m_i \text{Sym}^{2i-1}(\mathbb{C}^2)$ be two finite dimensional representations of SU_2 . For any representation E of $SU(2)$, let $E[i]$ denote the subspace of E on which S^1 , the standard maximal torus in $SU(2)$, operates by the character $z \rightarrow z^i$. Then we have the estimate,*

$$2 \dim(V \otimes W[1]) - \dim(V \otimes V[2] \oplus W \otimes W[2]) \leq \sum n_i^2 + \sum m_i^2.$$

Proof. The proof is an immediate consequence of the previous lemma on noting the well-known fact that for any representation π of SU_2 with trivial central character,

$$\dim \pi[0] - \dim \pi[2] = \dim \pi^{SU_2},$$

and therefore for representations V, W as above,

$$\begin{aligned} V \otimes V[0] - V \otimes V[2] &= \dim \text{End}_{SU_2}[V] = \sum_{i=0}^{i=d} n_i^2 \\ W \otimes W[0] - W \otimes W[2] &= \dim \text{End}_{SU_2}[W] = \sum_{i=1}^{i=d} m_i^2. \end{aligned}$$

\square

To get some feel for the dimensions involved, let's do an example. Let

$$\begin{aligned} V &= n_0 \cdot 1 \oplus n_2 \text{Sym}^2(\mathbb{C}^2) \oplus n_{2d} \text{Sym}^{2d}(\mathbb{C}^2), \\ W &= m_1 \text{Sym}^1(\mathbb{C}^2) \oplus m_{2r-1} \text{Sym}^{2r-1}(\mathbb{C}^2), \end{aligned}$$

assuming $2d > 2r - 1$. Then,

$$\begin{aligned} (V \otimes V)[0] &= n_0^2 + 3n_2^2 + (2d+1)n_{2d}^2 + 2n_0n_2 + 2n_0n_{2d} + 6n_2n_{2d}, \\ (W \otimes W)[0] &= 2m_1^2 + 2rm_{2r-1}^2 + 4m_1m_{2r-1}, \\ (V \otimes W)[1] &= n_0m_1 + n_0m_{2r-1} + 2n_2m_1 + 3n_2m_{2r-1} + 2n_{2d}m_1 + 2rn_{2d}m_{2r-1}. \end{aligned}$$

Further,

$$\begin{aligned} \dim(\text{End } V) &= n_0^2 + n_2^2 + n_{2d}^2 \\ \dim(\text{End } W) &= m_1^2 + m_{2r-1}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\dim(V \otimes V[0] \oplus W \otimes W[0]) - 2 \dim(V \otimes W[1]) \\ &= n_0^2 + 3n_2^2 + (2d+1)n_{2d}^2 + 2n_0n_2 + 2n_0n_{2d} + 6n_2n_{2d} + 2m_1^2 + 2rm_{2r-1}^2 + 4m_1m_{2r-1} \\ &\quad - 2(n_0m_1 + n_0m_{2r-1} + 2n_2m_1 + 3n_2m_{2r-1} + 2n_{2d}m_1 + 2rn_{2d}m_{2r-1}), \end{aligned}$$

which by Lemma 1 is always positive. On the other hand,

$$\begin{aligned} &\dim(V \otimes V[2] \oplus W \otimes W[2]) - 2 \dim(V \otimes W[1]) \\ &= 2n_2^2 + 2dn_{2d}^2 + 2n_0n_2 + 2n_0n_{2d} + 6n_2n_{2d} + 2m_1^2 + 2rm_{2r-1}^2 + 4m_1m_{2r-1} \\ &\quad - 2(n_0m_1 + n_0m_{2r-1} + 2n_2m_1 + 3n_2m_{2r-1} + 2n_{2d}m_1 + 2rn_{2d}m_{2r-1}), \end{aligned}$$

is ‘usually’ positive but not always, and something which we are not able fully understand. We may want to impose an auxiliary condition such as $\dim V - \dim W = 1$.

2.

Rest of this paper is written with the aim of improving upon the inequality offered by Lemma 1 in the case of different central characters, or equivalently to understand given the representation V of $SU(2)$, what are the choices for the representation W so that the difference of the left hand side of the inequality in Lemma 1 with the right hand side is as small as possible; we do this by analyzing the argument which goes into the proof of the lemma more carefully.

Write,

$$\begin{aligned} V &= \sum_{i=0}^{i=d} n_i \text{Sym}^{2i}(\mathbb{C}^2), \\ W &= \sum_{i=1}^{i=d} m_i \text{Sym}^{2i-1}(\mathbb{C}^2). \end{aligned}$$

We use Clebsch-Gordon theorem to expand the character $\chi_1 \text{St}_n - \chi_2 \text{St}_{n+1}$ out:

$$\begin{aligned} \chi_1 \text{St}_n - \chi_2 \text{St}_{n+1} &= \sum_{i=0}^{i=d} n_i \text{St}_{2i} \text{St}_n - \sum_{i=1}^{i=d} m_i \text{St}_{2i-1} \text{St}_{n+1} \\ &= \sum_{i=0}^{i=d} n_i (\text{St}_{n+2i} + \cdots + \text{St}_{n-2i}) - \sum_{i=1}^{i=d} m_i (\text{St}_{n+2i} + \cdots + \text{St}_{n-2i+2}) \\ &= n_0 \text{St}_n + \sum_{i=1}^{i=d} (n_i - m_i) (\text{St}_{n+2i} + \cdots + \text{St}_{n-2i}) + \sum_{i=1}^{i=d} m_i \text{St}_{n-2i} \\ &\stackrel{*}{=} (n_d - m_d) \text{St}_{n+2d} + (n_d - m_d + n_{d-1} - m_{d-1}) \text{St}_{n+2d-2} + \cdots \\ &\quad + (n_d - m_d + n_{d-1} - m_{d-1} + n_{d-2} + \cdots + n_1 - m_1) \text{St}_{n+2} \\ &\quad + (n_d - m_d + n_{d-1} - m_{d-1} + n_{d-2} + \cdots + n_1 - m_1 + n_0) \text{St}_n \\ &\quad + (n_d - m_d + n_{d-1} - m_{d-1} + n_{d-2} + \cdots + n_1 - m_1 + m_1) \text{St}_{n-2} \\ &\quad + \cdots + (n_d - m_d + m_d) \text{St}_{n-2d}. \end{aligned}$$

Using the inequality,

$$a^2 + b^2 \geq \frac{1}{2}(a - b)^2,$$

and grouping the first term in the above summation with the last, then the second with the second last etc. (which leaves the middle term alone, which we do not consider), we get:

$$\int_{SU_2} [\chi_1 \text{St}_n - \chi_2 \text{St}_{n-1}]^2 \geq \frac{1}{2}(m_1^2 + \cdots + m_d^2).$$

Similarly, using the same inequality,

$$a^2 + b^2 \geq \frac{1}{2}(a - b)^2,$$

but now grouping the first term in the equation (\star) with the second last term, then the second with the third last etc., we get:

$$\int_{SU_2} [\chi_1 St_n - \chi_2 St_{n-1}]^2 \geq \frac{1}{2}(n_0^2 + n_1^2 + \cdots + n_{d-1}^2 + 2n_d^2).$$

We record this in the following lemma improving Lemma 1.

Lemma 2. *Let $V = \sum_{i=0}^{i=d} n_i \text{Sym}^{2i}(\mathbb{C}^2)$, and $W = \sum_{i=1}^{i=d} m_i \text{Sym}^{2i-1}(\mathbb{C}^2)$ be two finite dimensional representations of SU_2 . Assume that the character of V is χ_1 and that of W is χ_2 . Then for any large enough integer n , we have the estimate,*

$$\int_{SU_2} [\chi_1 St_n - \chi_2 St_{n-1}]^2 \geq \frac{1}{2} \max\{(m_1^2 + \cdots + m_d^2), (n_0^2 + n_1^2 + \cdots + n_{d-1}^2 + 2n_d^2)\},$$

and hence,

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) - 2 \dim(V \otimes W[1]) \geq \frac{1}{2} \max\{(m_1^2 + \cdots + m_d^2), (n_0^2 + n_1^2 + \cdots + 2n_d^2)\}.$$

3.

We analyze a little further the expression above for $\chi_1 St_n - \chi_2 St_{n+1}$ in equation (\star) in the hope of finding an inequality in the other direction. We have,

$$\chi_1 St_n - \chi_2 St_{n+1} = a_d St_{n+2d} + \cdots + a_1 St_{n+2} + (a_1 + n_0) St_n + (a_1 + m_1) St_{n-2} + \cdots + (a_d + m_d) St_{n-2d},$$

where

$$a_i = \sum_{j \geq i} (n_j - m_j).$$

Therefore,

$$\begin{aligned} \int_{SU_2} [\chi_1 St_n - \chi_2 St_{n+1}]^2 &= (a_1 + n_0)^2 + \sum_{i \geq 1} [a_i^2 + (a_i + m_i)^2] \\ &= (a_1 + n_0)^2 + \sum_{i \geq 1} 2a_i(a_i + m_i) + \sum_{i \geq 1} m_i^2. \end{aligned}$$

It follows that the L^2 -norm of $\chi_1 St_n - \chi_2 St_{n+1}$ is controlled by the signs of $a_i(a_i + m_i)$, and among all cases, a favorable case would be when all the $a_i(a_i + m_i)$ have the same sign. This does happen in an interesting set of cases which is what we analyse next.

For a representation,

$$E = \sum_{i=0}^{i=d} \ell_i \text{Sym}^i(\mathbb{C}^2),$$

let E^- denote the representation,

$$E^- = \sum_{i=0}^{i=d} \ell_i \text{Sym}^{i-1}(\mathbb{C}^2),$$

with the understanding that $\text{Sym}^{-1}(\mathbb{C}^2) = 0$.

Write the representations $V = \sum_{i=0}^{i=d} n_i \text{Sym}^{2i}(\mathbb{C}^2)$, and $W = \sum_{i=1}^{i=d} m_i \text{Sym}^{2i-1}(\mathbb{C}^2)$, in terms of partitions as:

$$V = \underbrace{2d+1, \dots, 2d+1}_{n_d}, \underbrace{2d-1, \dots, 2d-1}_{n_{d-1}}, \dots, \underbrace{3, \dots, 3}_{n_1}, \underbrace{1, \dots, 1}_{n_0},$$

and

$$W = \underbrace{2d, \dots, 2d}_{m_d}, \underbrace{2d-2, \dots, 2d-2}_{m_{d-1}}, \dots, \underbrace{2, \dots, 2}_{m_1}.$$

The corresponding partitions associated to V^- and W^- are:

$$V^- = \underbrace{2d, \dots, 2d}_{n_d}, \underbrace{2d-2, \dots, 2d-2}_{n_{d-1}}, \dots, \underbrace{2, \dots, 2}_{n_1},$$

and

$$W^- = \underbrace{2d-1, \dots, 2d-1}_{m_d}, \underbrace{2d-3, \dots, 2d-3}_{m_{d-1}}, \dots, \underbrace{1, \dots, 1}_{m_1}.$$

On the set of partitions $\underline{a} = a_1 m + a_2(m-1) + a_3(m-1) + \dots + a_m 1$, there is a partial ordering defined by $\underline{a} \geq_Y \underline{b}$ for $\underline{b} = b_1 m + b_2(m-1) + b_3(m-1) + \dots + b_m 1$, if $a_1 + \dots + a_t \geq b_1 + \dots + b_t$ for all integers t . This partial order \geq_Y on the set of all partitions is called the *Young order* giving rise to *Young lattice*. It is different from the much more used partial order on the set of all partitions which is called the *Dominance order*. The Young order has the property that $\underline{a} \geq_Y \underline{b}$ if and only if if we were to represent \underline{a} and \underline{b} as Young tableaux, then \underline{b} sits inside \underline{a} as a sub-tableaux, i.e., \underline{b} is obtained from \underline{a} by removing some boxes from the end of each row.

An important property of Young order is that if \underline{a} is a partition of $n+1$ corresponding to an irreducible representation $\pi_{\underline{a}}$ of S_{n+1} , and \underline{b} is a partition of n corresponding to an irreducible representation $\pi_{\underline{b}}$ of S_n , then the representation $\pi_{\underline{a}}$ of S_{n+1} contains the representation $\pi_{\underline{b}}$ of S_n when restricted to S_n if and only if $\underline{a} \geq_Y \underline{b}$. (Although this note is motivated by questions on branching laws from $\text{GL}(n+1)$ to $\text{GL}(n)$ in the p-adic context, it is not clear if Young order, and its relation to branching law from S_{n+1} to S_n is really relevant to this note, or it is a red herring.)

Therefore, the condition $W \geq_Y V^-$ means,

$$\begin{aligned} m_d &\geq n_d, \\ m_d + m_{d-1} &\geq n_d + n_{d-1}, \\ m_d + m_{d-1} + m_{d-2} &\geq n_d + n_{d-1} + n_{d-2}, \\ &\dots \geq \dots, \\ m_d + m_{d-1} + m_{d-2} + \dots + m_1 &\geq n_d + n_{d-1} + n_{d-2} + \dots + n_1. \end{aligned}$$

These conditions can be reinterpreted in terms of a_i defined earlier as $a_i = \sum_{j \geq i} (n_j - m_j)$ to simply $a_i \leq 0$ for all $i \geq 1$.

The condition $V \geq_Y W^-$ means,

$$\begin{aligned} n_d &\geq 0 \\ n_d + n_{d-1} &\geq m_d, \\ n_d + n_{d-1} + n_{d-2} &\geq m_d + m_{d-1}, \\ n_d + n_{d-1} + n_{d-2} + n_{d-3} &\geq m_d + m_{d-1} + m_{d-2}, \\ \dots &\geq \dots, \\ n_d + n_{d-1} + n_{d-2} + \dots + n_1 &\geq m_d + m_{d-1} + m_{d-2} + \dots + m_2, \end{aligned}$$

which can be rewritten as $a_i + m_i \geq 0$ for all $i \geq 1$.

Thus, it follows that for representations V, W with $W \geq_Y V^-$ and $V \geq_Y W^-$, we have,

$$a_i(a_i + m_i) \leq 0.$$

Thus we have proved the following proposition.

Proposition 1. *Let $V = \sum_{i=0}^d n_i \text{Sym}^{2i}(\mathbb{C}^2)$, and $W = \sum_{i=1}^d m_i \text{Sym}^{2i-1}(\mathbb{C}^2)$ be two finite dimensional representations of SU_2 . Then for $W \geq_Y V^-$ and $V \geq_Y W^-$, and for n large enough, we have the estimate,*

$$\int_{SU_2} [\chi_1 \text{St}_n - \chi_2 \text{St}_{n-1}]^2 \leq (m_1^2 + \dots + m_d^2) + (a_1 + n_0)^2,$$

and therefore,

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) - 2 \dim(V \otimes W[1]) \leq (m_1^2 + \dots + m_d^2) + (a_1 + n_0)^2.$$

4.

We end the note by discussing an example.

Example: Let

$$\begin{aligned} V &= \sum_{i=0}^d (n_i + m_i) \text{Sym}^i(\mathbb{C}^2), \\ W &= \sum_{i=0}^d n_i \text{Sym}^{i-1}(\mathbb{C}^2) + \sum_{i=0}^d m_i \text{Sym}^{i+1}(\mathbb{C}^2), \end{aligned}$$

with character of V to be χ_1 and character of W to be χ_2 . It can be seen that for these two representations of SU_2 , the conditions in Proposition 1, i.e., $W \geq V^-$ and $V \geq W^-$, are satisfied.

Since,

$$\begin{aligned} \text{St}_i \text{St}_n - \text{St}_{i+1} \text{St}_{n+1} &= [\text{St}_{n+i} + \dots + \text{St}_{n-i}] - [\text{St}_{n+i+2} + \dots + \text{St}_{n-i}] \\ &= -\text{St}_{n+i+2}, \end{aligned}$$

and similarly,

$$\begin{aligned} \text{St}_i \text{St}_n - \text{St}_{i-1} \text{St}_{n+1} &= [\text{St}_{n+i} + \dots + \text{St}_{n-i}] - [\text{St}_{n+i} + \dots + \text{St}_{n-i+2}] \\ &= \text{St}_{n-i}. \end{aligned}$$

it follows that,

$$\begin{aligned}\chi_1 \text{St}_n - \chi_2 \text{St}_{n+1} &= \sum n_i (\text{St}_i \text{St}_n - \text{St}_{i-1} \text{St}_{n+1}) + \sum m_i (\text{St}_i \text{St}_n - \text{St}_{i+1} \text{St}_{n+1}) \\ &= \sum n_i \text{St}_{n-i} - \sum m_i \text{St}_{n+i+2}.\end{aligned}$$

Therefore for n large enough,

$$\int_{\text{SU}(2)} [\chi_1 \text{St}_n - \chi_2 \text{St}_{n+1}]^2 = \sum (n_i^2 + m_i^2),$$

and therefore,

$$\dim(V \otimes V[0] \oplus W \otimes W[0]) - 2 \dim(V \otimes W[1]) = \sum (n_i^2 + m_i^2).$$

Remark : Putting $n_i = m_i$ in this example, we find that in a certain sense, Lemma 2 is the best possible.

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